Discrete mathematics

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1 Introduction

This short paper is almost entirely based on a section in the introductory book "Discrete Mathematics" by Martin Aigner (American Mathematical Society, 2007), pages 8-10.

2 Elementary counting principles

An **incidence system** (S, T, I) consists of two sets S and T and a relation I (called the incidence) between the elements of S and T.

If there is a relation $a\ I\ b$ between $a \in S$ and $b \in T$, then we call a and b *incident*.

2.1 Rule of Double Counting

Let **incidence system** (S, T, I) be an incidence system, and for $a \in S$ let $r(a)$ denote the number of elements of T incident to a, and analogously, $r(b)$ denote the number of elements of S incident to b. Then the following relation holds:

$$\sum_{a \in S} r(a) = \sum_{b \in T} r(b) \tag{1}$$

It becomes clear at once that this rule is valid if we consider the incidence system as a rectangular array. We number the elements of S and T thus: $S = \{a_1, \ldots, a_m\}$, $T = \{b_1, \ldots, b_n\}$. We now create an $m \times n$ matrix $M = (m_{ij})$, called the **incidence matrix**, by setting

$$m_{ij} = \begin{cases} 1 & \text{if } a_i\ I\ b_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The value of $r(a_i)$ is then precisely the number of ones in the ith row, and analogously, $r(b_j)$ is the number of ones in the jth column. The sum $\sum_{i=1}^{m} r(a_i)$ is then equal to the total number of ones (counted row-wise), while $\sum_{j=1}^{n} r(b_j)$ gives the same number (counted column-wise).

3 Example: the divisor function

Consider the numbers from 1 to 4, with $S = T = \{1, ..., 4\}$, and say that $i \in S$, $j \in T$ are incident if i is a divisor of j.

The associated incidence matrix then has the following form

	1	2	3	4
1	1	1	1	1
2		1		1
3			1	
4				1

The number of ones in column j is equal to the number of divisors of j, which we denote by $t(j)$. For example, $t(2) = 2$, $t(4) = 3$.

We now pose the question, how many divisors does a number between 1 and 4 have on *average*? That is, we would like to compute

$$\bar{t}(4) = \frac{1}{4}\sum_{j=1}^{4} t(j). \tag{3}$$

In our example, $\bar{t}(4) = 2$. From the table we obtain the following values:

n	1	2	3	4
$\bar{t}$(n)	1	$\frac{3}{2}$	$\frac{5}{3}$	2

How big, then, is $\bar{t}(n)$ for arbitrary n? At first glance, the prospects of answering this question seem hopeless. For prime numbers p, we have $t(p) = 2$, while for a power of two, the value $t(2^k) = k + 1$ results.[1]

Nonetheless, let us attempt to apply the rule of double counting. Counting by columns, we obtain, as we have seen, $\sum_{j=1}^{n} t(j)$. How many ones are there in the ith row? Clearly, the ones represent the multiples of i, namely $1 \cdot i$, $2 \cdot i$, $\ldots$, and the last multiple less than or equal to n is $\lfloor \frac{n}{i} \rfloor \cdot i$, and so $r(i) = \lfloor \frac{n}{i} \rfloor$.

Our rule therefore yields

$$\bar{t}(n) = \frac{1}{n}\sum_{j=1}^{n} t(j) = \frac{1}{n}\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor \sim \frac{1}{n}\sum_{i=1}^{n} \frac{n}{i} = \sum_{i=1}^{n} \frac{n}{i}, \tag{4}$$

where the error in passing from $\lfloor \frac{n}{i} \rfloor$ to $\frac{n}{i}$ for all i is less than 1, and is so in the sum as well.[2]

We shall often encounter the last value, $\sum_{i=1}^{n} \frac{n}{i}$. It is called the nth **harmonic number** H_n. From calculus we know that $H_n \sim \log n$ is about the same size of the natural logarithm.[3]

Thus we obtain the astounding result that the divisor function, despite its irregularity, behaves completely regularly *on average*, namely, $\bar{t}(n) \sim \log n$.

[1] For a k-power of 2, every preceding power of 2 will account for k divisors, while 1 is a divisor for every integer; hence the $k + 1$

[2] By the definition of the floor function, $| \lfloor n \rfloor - n | < 1$.

[3] The harmonic number can be interpreted as a Riemann sum of the integral of the inverse function, which is by definition the natural logarithm.